Recitation 3. March 16

Focus: nullspaces, systems of equations, dimension and rank, orthogonal subspaces, projection matrices

The **nullspace** of an $m \times n$ matrix A is the vector space consisting of those $\boldsymbol{v} \in \mathbb{R}^n$ such that $A\boldsymbol{v} = 0$. To find a **basis** of the nullspace, put the matrix in reduced row echelon form, and look at the **pivot and free variables**.

The general solution to a system of equations $A\boldsymbol{v} = \boldsymbol{b}$ is:

$$\boxed{\boldsymbol{v} = \boldsymbol{v}_{\text{particular}} + \boldsymbol{w}_{\text{general}}}$$

where $\boldsymbol{v}_{\text{particular}}$ is a particular solution, and $\boldsymbol{w}_{\text{general}}$ is a general element of $N(A)$.

The **dimension** of a vector space is the number of vectors in a basis (i.e. a collection of linearly independent vectors which span the vector space in question). The **rank** of a matrix A is the dimension of its column space $C(A)$.

Two subspaces V, W of $\mathbb{R}^n$ are called **orthogonal** if any vector in a basis of V is orthogonal (a.k.a. perpendicular, a.k.a. has dot product 0) to any vector in a basis of W. For any matrix A:

- the **column space** is the orthogonal complement of the **left nullspace**
- the **row space** is the orthogonal complement of the **nullspace**

The projection of a vector $\boldsymbol{b} \in \mathbb{R}^n$ onto a subspace $V \subset \mathbb{R}^n$ is the closest vector $\boldsymbol{p} \in V$ to $\boldsymbol{b}$. It can be computed by:

$$\boxed{\boldsymbol{p} = \underbrace{A(A^T A)^{-1} A^T}_{\text{projection matrix}} \boldsymbol{b}}$$

for any matrix A with column space V. The columns of A must be linearly independent to apply the formula above!

1. Use Gauss-Jordan elimination to compute the null space $N(X)$ of the matrix

$$X = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 3 & -2 & 0 & 5 \\ -2 & 4 & -1 & -5 \end{bmatrix}$$

Then find the general solution to the system of equations:

$$X\boldsymbol{v} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

Solution:

Solution:

2. Find bases for the four fundamental subspaces of the matrix X in Question 1, and check that these four subspaces are orthogonal complements of each other, in the appropriate pairs.

Solution:

Solution:

3. Consider the subspace V with basis given by $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Compute the closest point of V to the vector $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$. Check that the projection matrix P_V onto the subspace V satisfies that $P_V^2 = P_V$

Solution: